Efficient OAT* designs

J.-M. Fédou, G. Menez, L. Pronzato, M.-J. Rendas

I3S Laboratory, CNRS/UNS (Sophia Antipolis, France)

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*One At a Time

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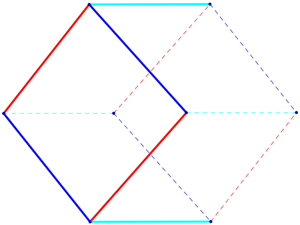
Problem

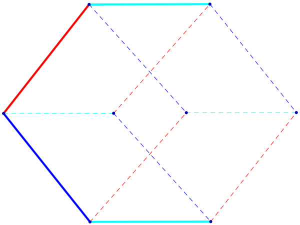
Find subgraphs $G \subset Q_d$ of the d-dimensional hypercube with the property:

$\forall i \in \{1, \ldots, d\}$, the number of edges of G joining nodes that **differ only in the i-th coordinate** is equal to m.

We say that graphs with this property are (d, m)-equitable.



$(3, 2)$-equitable



Not $(3, m)$-equitable

Q_3

Motivation

Morris **elementary effects** screening method for **sensitivity analysis** (Technometrics, 1991)

Commonly used screening method for analysis of $f : \mathbb{R}^d \to \mathbb{R}$

- Partitions input factors into *linear*, *negligible* and *non-linear/mixed*
- Makes no assumptions about f
- Simple (linear in the number of inputs), OAT global method.

Based on statistical analysis of

Elementary effect along direction $i \in \{q, \ldots, d\}$

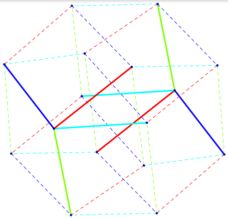
$$d_i(y) \triangleq \frac{1}{\Delta} [f(y + \Delta e_i) - f(y)] \,, \quad i \in \{1, \ldots, d\}$$

Link to our work

Morris clustered designs

Design matrices B that allow computation of $m > 1$ elementary effects along each direction (i.e., each evaluation of f is used to compute several d_i's).

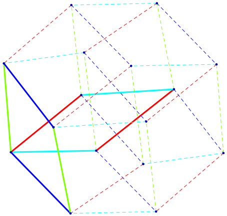
$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$



10 points in Q_4

(4, 2)-equitable subgraphs

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



7 points in Q_4

Why coming back to the problem?

Limitations of Morris clustered construction

- not guided by m
- does not yield all possible values of m
- minimality of the size of the designs (efficiency) is not guaranteed.
- factored version (the most efficient) defined only when d is not prime
- not always *equitable*

Our contribution

Constructive algorithm for generation of the clustered designs of Morris method guided by the target value of m and the dimension d of the input space

- Handles generic values of (d, m).
- Always leads to equitable designs.
- For pairs (d, m) for which Morris construction is defined, leads to designs of the same complexity.

How do we do it?

Two basic ideas

1. (d, m)-equitable subgraphs are recursively generated, by combining smaller equitable solutions (for smaller values of d and m)
2. use a polynomial representation to manipulate subgraphs and prove their properties

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Polynomial representation of subgraphs of Q_d

Coding points of Q_d by monomials

$$s = \{s_1, s_2, \dots, s_d\} \longrightarrow \mathcal{P}_s(X_1, X_2, \dots, X_d) = X_1^{s_1} X_2^{s_2} \dots X_d^{s_d}$$

Example

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \in Q_5 \to X_2 X_3 X_5 \quad \in K(X_1, \dots, X_5) = K_5$$

Coding subgraphs of Q_d by polynomials

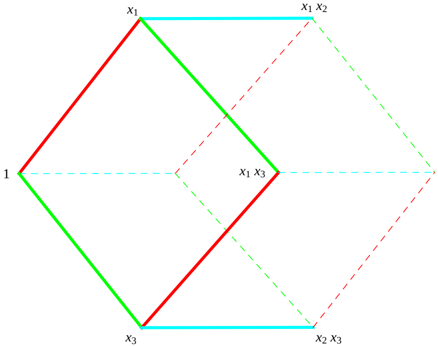
$$G \subset Q_d \to \mathcal{P}_G = \sum_{s \in G} \mathcal{P}_s$$

$\mathcal{P}_G$: degree at most one in each variable, coefficients in $\{0, 1\}$.

Polynomial representation of subgraphs of Q_d

Example

$$P = 1 + x_1 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 \subset Q_3$$



Edge coloring of Q_3:

- : x_1
- : x_2
- : x_3

Polynomial representation of subgraphs of Q_d

Scalar product and structure

Definition of $\langle \cdot, \cdot \rangle$

$\mathcal{P}_s$, $\mathcal{P}_{s'}$ two monomials ($s, s' \in Q_d$)
Define the scalar product

$$\langle \mathcal{P}_s, \mathcal{P}_{s'} \rangle = 1_{s=s'} \quad .$$

Extension to polynomials ($G, G' \subset Q_d$)

$$\langle \mathcal{P}_G, \mathcal{P}_{G'} \rangle = \sum_{s \in G, s \in G'} \langle \mathcal{P}_s, \mathcal{P}_{s'} \rangle \quad .$$

Example

$$\langle X_1 X_2, X_1 X_2 \rangle = 1, \qquad \langle X_1 X_2, X_1 X_2 X_3 \rangle = 0$$

$$\langle 1 + X_1 + X_2 + X_1 X_2, 1 + X_1 X_2 + X_3 \rangle = 2$$

Properties

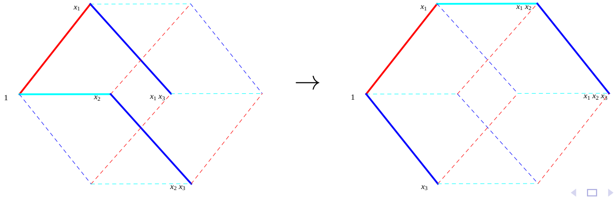
- $\langle P_G, P_{G'} \rangle = |G \cap G'|$
- $\langle P_G, P_G \rangle = |G|$

Algebra over the polynomials

- Addition $\Leftrightarrow$ graph union (nodes multiplicity may be > 1)
- Multiplication is defined modulo $X_i^2 = 1, i \in \{1, \ldots, d\}$
  Multiplication of P_G by a monomial $s = X_i \Leftrightarrow$ reflection of G along edge i

Example (X_1 corresponds to red edges)

$$\begin{aligned} X_1(1 + X_1 + X_2 + X_1X_3 + X_2X_3) &= X_1 + X_1^2 + X_1X_2 + X_1^2X_3 + X_1X_2X_3 \\ &= X_1 + 1 + X_1X_2 + X_3 + X_1X_2X_3 \end{aligned}$$



Problem reformulation in terms of polynomials

Facts:

1. edges of color i are preserved by multiplication by X_i. All other edges are moved elsewhere in Q_d
2. (remember that $|G \cap G'| = \langle P_G, P_{G'} \rangle$)
3. $\Rightarrow$ the number of edges of G of color i is exactly $2 \langle P_G, X_i P_G \rangle$

Problem reformulation

Optimal (d, m)-equitable designs are the solutions of

$$P^\star = \arg\min_{P \in K_d} \langle P, P \rangle$$

$$\text{s.t. } \langle P^\star, X_i P^\star \rangle = 2m, \qquad i \in \{1, 2, \ldots, d\}.$$

We drop minimality, and assess the simpler problem of finding small (d, m)-equitable designs (not necessarily minimal).

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Generation of (d, m)-equitable subgraphs of Q_d

Recursive (in m) algorithm

Initialisation

- $m = 1$, generic d

$$G_d^1 = 1 + \sum_{i=1}^{d} X_1 \cdots X_i \quad .$$

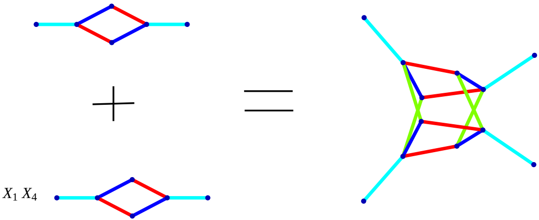
Generation of (d, m)-equitable subgraphs of Q_d

Induction

- m even

$$G_d^m = G_{d-1}^{\frac{m}{2}} + X_1 X_d G_{d-1}^{\frac{m}{2}}$$

Example: $G_4^4 = G_3^2 + X_1 X_4 G_3^2$



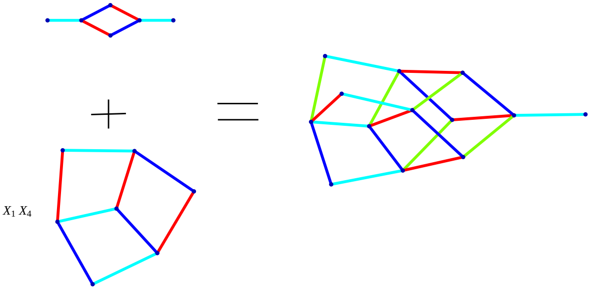
Generation of (d, m)-equitable subgraphs of Q_d

Induction

- m odd

$$G_d^m = G_{d-1}^{\frac{m-1}{2}} + X_1 X_d \, G_{d-1}^{\frac{m+1}{2}}$$

Example: $G_4^5 = G_3^2 + X_1 X_4 \, G_3^3$



Theorem

G_d^m are (d, m)-equitable

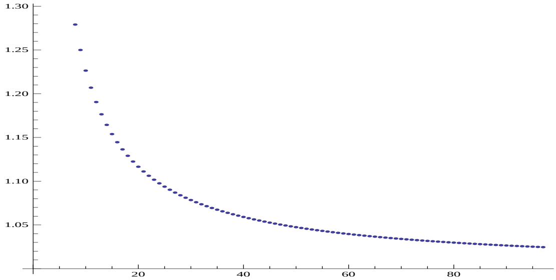
Proof: use properties of scalar product (assumes an additional condition of solutions for consecutive values of m)

Generation of (d, m)-equitable subgraphs of Q_d

Economy

Morris index, ($|G_d^m|$ should be small $\Leftrightarrow \chi$ large)

$$\text{Economy: } \chi = \frac{\text{total \# elementary effects}}{|G_d^m|} = \frac{md}{|G_d^m|}$$



Evolution of χ as d grows, $m = 10$.

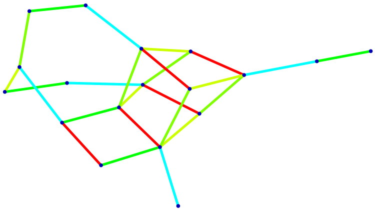
Deceptive behavior (asymptotically, no increased efficiency compared to random placement of m stars).

Generation of (d, m)-equitable subgraphs of Q_d

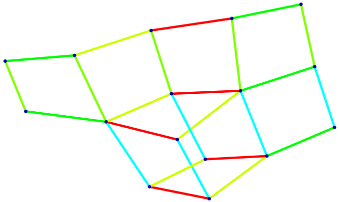
Topology and Initalisation

Other families of solutions can be obtained, by changing the initialization for small values of m

This has an impact on the topology (and on the complexity!!) of the resulting designs



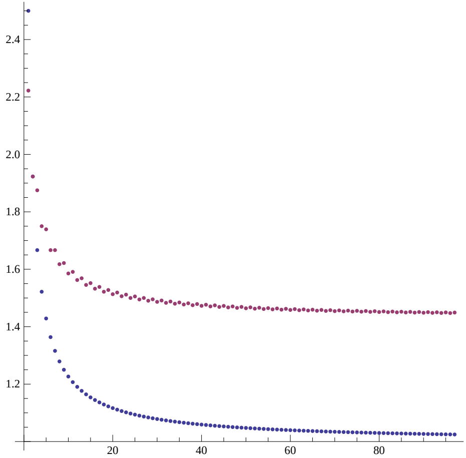
G_5^5, Init $m = 1$ only



G_5^5, Init $m = 2, 3$

Generation of (d, m)-equitable subgraphs of Q_d

Economy



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Factored (d, m)-equitable designs

Direct application of our algorithm leads to less efficient designs than Morris when these are defined.

Factored application of our generic solution

$$q_{\min}(m) \triangleq \lceil \log_2(m) \rceil + 1 \ ,$$

$$d = (c-1)q_{\min}(m) + r, \qquad r \in \{q_{\min}(m), \dots, 2q_{\min}(m) - 1\} \ .$$

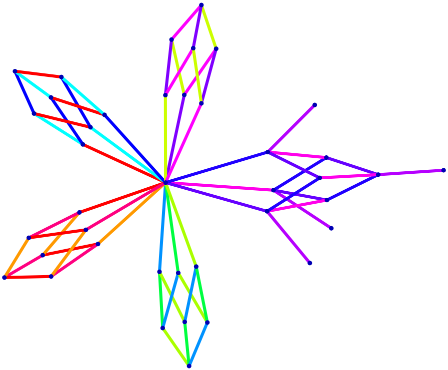
$$G_{Morris}(d,m) = G\left(q_{\min}, m\right) + \sum_{j=1}^{c-2} \left(\text{Shift}_{jq_{\min}} G\left(q_{\min}, m\right) - 1\right) + \text{Shift}_{(c-1)q_{\min}} G(r,m)$$

Fully-defined and provably equitable version of the basic idea of Morris factored designs.

Factored (d, m)-equitable designs

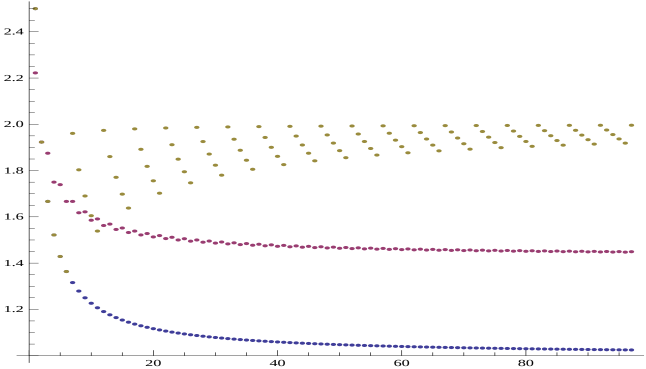
Example

$\mathbf{G^4_{17}}$: 4 complete Q_3 ($X_1 \cdots X_3$, $X_4 \cdots X_6$, $X_7 \cdots X_9$, $X_{10} \cdots X_{12}$), together with G^4_5 (over $X_{13} \cdots X_{17}$)



Factored (d, m)-equitable designs

Complexity



Evolution of χ as d grows, $m = 10$.

Factored designs, original designs with init G_d^1, and with init G_d^2, G_d^3.

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Summary

1. recursive algorithm for (d, m)-equitable graphs that completes the definition of clustered Morris designs
2. uses polynomial representation of subgraphs of the hypercube and an appropriate definition of inner product as formal tools.

Further work

Pending issues ...

- minimality (of factored designs) ?
- effect of initialization ?
- relation to other classes of subgraphs of the hypercube (median graphs, mesh graphs,...)?
- relation to LHS : properties of reduction to a proper subspace of the input space?
- Extend analysis to consider clustered computation of “elementary cross-effects” (some preliminary results).

Generation of (d, m)-equitable subgraphs of Q_d

Demonstration (equitable designs)

m even. Assume $G_{d-1}^{m/2}$ is $(d-1, m)$-equitable.

$$\left\langle G_m^d, X_i G_m^d \right\rangle = \begin{cases} \left\langle G_{d-1}^{\frac{m}{2}}, X_i G_{d-1}^{\frac{m}{2}} \right\rangle + \\ \qquad \left\langle X_1 X_d G_{d-1}^{\frac{m}{2}}, X_i X_1 X_d G_{d-1}^{\frac{m}{2}} \right\rangle = 2m, & \text{if } i < d \\ \left\langle G_{d-1}^{\frac{m}{2}}, X_1 G_{d-1}^{\frac{m}{2}} \right\rangle + \\ \qquad \left\langle X_1 X_d G_{d-1}^{\frac{m}{2}}, X_1 G_{d-1}^{\frac{m}{2}} \right\rangle = 2m, & \text{if } i = d \end{cases} .$$

Generation of (d, m)-equitable subgraphs of Q_d

Demonstration (equitable designs)

m odd. Assume $G_{d-1}^{\frac{m-1}{2}}$ and $G_{d-1}^{\frac{m+1}{2}}$ equitable

$$\langle G_d^m, X_i G_d^m \rangle = \begin{cases} \left\langle G_{d-1}^{\frac{m-1}{2}}, X_i G_{d-1}^{\frac{m-1}{2}} \right\rangle + \\ \quad + \left\langle G_{d-1}^{\frac{m+1}{2}}, X_i G_{d-1}^{\frac{m+1}{2}} \right\rangle, & \text{if } i < d \\ 2 \left\langle G_{d-1}^{\frac{m-1}{2}}, X_1 G_{d-1}^{\frac{m+1}{2}} \right\rangle, & \text{if } i = d \end{cases}$$

$$= \begin{cases} (m-1) + (m+1) = 2m, & \text{if } i < d \\ 2 \left\langle G_{d-1}^{\frac{m-1}{2}}, X_1 G_{d-1}^{\frac{m+1}{2}} \right\rangle, & \text{if } i = d \end{cases}$$

Thus

$$G_d^m \text{ is } (d, m)\text{-equitable } \Leftrightarrow \left\langle G_{d-1}^{\frac{m-1}{2}}, X_1 G_{d-1}^{\frac{m+1}{2}} \right\rangle = m$$

It can be shown that

$$\left\langle G_{d-1}^{k-1}, X_1 G_{d-1}^{k} \right\rangle = 2k-1 \Rightarrow \left\langle G_d^{2k-1}, X_1 G_d^{2k} \right\rangle = 4k-1$$
$$\left\langle G_{d-1}^{k}, X_1 G_{d-1}^{k+1} \right\rangle = 2k+1 \Rightarrow \left\langle G_d^{2k}, X_1 G_d^{2k+1} \right\rangle = 4k+1$$

Generation of (d, m)-equitable subgraphs of Q_d

Demonstration

$$\left\langle G_{d-1}^{k}, X_1 G_{d-1}^{k+1} \right\rangle = 2k+1$$

Check that is true for $k = 1$, using the construction G_d^2.

$$\begin{aligned}\left\langle G_d^1, X_1 G_d^2 \right\rangle &= \left\langle (1 + \sum_{i=1}^{d} X_1 \cdots X_i), (X_1 + X_d)(1 + \sum_{j=1}^{d-1} X_1 \cdots X_j) \right\rangle \\ &= \langle 1, 1 \rangle + \langle X_1, X_1 \rangle + \langle X_1 \cdots X_d, X_1 \cdots X_d \rangle \\ &= 3\end{aligned}$$

The identity is thus valid for all k, completing the proof that our algorithm generates (d, m)-equitable subgraphs of Q_d.

Morris designs

$$\mathbb{R}^d = \prod_{j=1}^{t} \mathbb{R}^q, \qquad d = tq \qquad Y = \bigcup_{j=1}^{t} Y^j,$$

where

$$Y^j = v_j + C\,[\underbrace{O_q \;\cdots\; O_q}_{j-1 \text{ blocks}} \;\; I_q \;\; \underbrace{O_q \;\cdots O_q}_{t-j \text{ blocks}}], \qquad j = 1, \ldots, t \;\;,$$

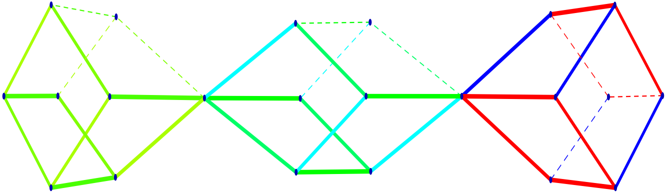
$$B_M = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ C & O & O & \cdots & O \\ J & C & O & \cdots & O \\ J & J & C & \cdots & O \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ J & J & J & \cdots & C \end{bmatrix}$$

0: q-element (row) vector of zeros, J: $n_C \times q$ matrix of ones.

Morris designs

$d = 9 = 3 \times 3$

$\begin{bmatrix} C & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} J & C & 0 \end{bmatrix}$ $\begin{bmatrix} J & J & C \end{bmatrix}$



$\{X_1 \cdots X_3\}$ $\{X_4 \cdots X_6\}$ $\{X_7 \cdots X_9\}$

Morris designs

Choice of C

Chose $\mathcal{I} \subset \{1, \dots, q\}$. Let the rows of C(of dimension $n_C \times q$) be the set of all binary vectors with ℓ entries equal to one, $\forall \ell \in \mathcal{I}$.

$$n_C = \sum_{\ell \in \mathcal{I}} C_\ell^q$$

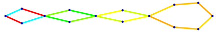
$$m(\mathcal{I}) = I(1)I(q) + \sum_{j=2}^{q} I(j-1)I(j)C_{j-1}^{q-1}$$

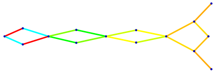
Size of Morris designs

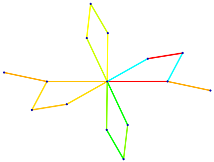
$$n_M = tn_C + 1 = \frac{d}{q} \sum_{\ell \in \mathcal{I}} C_\ell^q + 1$$

Initialisation

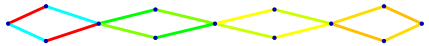
$m = 2$ d odd

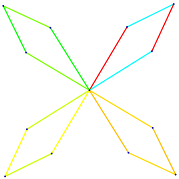






$m = 2,$ d even





Initialisation

$m = 3$