

A new Linear Time Bi-level $\ell_{1,\infty}$ projection ; Application to the sparsification of auto-encoders neural networks

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Abstract—The $\ell_{1,\infty}$ norm is an efficient-structured projection, but the complexity of the best algorithm is, unfortunately, $\mathcal{O}(nm \log(nm))$ for a matrix in $\mathbb{R}^{n \times m}$.

In this paper, we propose a new bi-level projection method, for which we show that the time complexity for the $\ell_{1,\infty}$ norm is only $\mathcal{O}(nm)$ for a matrix in $\mathbb{R}^{n \times m}$.

Moreover, we provide a new $\ell_{1,\infty}$ identity with mathematical proof and experimental validation.

Experiments show that our bi-level $\ell_{1,\infty}$ projection is $\mathcal{O}(\log(nm))$ times faster than the actual fastest algorithm. Our bi-level $\ell_{1,\infty}$ projection outperforms the sparsity of the usual $\ell_{1,\infty}$ projection while keeping the same or slightly better accuracy in classification applications.

I. INTRODUCTION

Sparsity requirement appears in many machine learning applications, such as the identification of biomarkers in biology [1]. It is well known that the impressive performance of neural networks is achieved at the cost of a high-processing complexity and large memory requirement [2]. Recently, advances in sparse recovery and deep learning have shown that training neural networks with sparse weights not only improves the processing time, but most importantly improves the robustness and test accuracy of the learned models [3], [4], [5][6], [7]. Regularizing techniques have been proposed to sparsify neural networks, such as the popular *LASSO* method [8], [9]. The *LASSO* considers the ℓ_1 norm as Lagrangian regularization. Group-*LASSO* originally proposed in [10], was used in order to sparsify neural networks without loss of performance [11], [12], [13]. Unfortunately, the classical Group-*LASSO* algorithm is based on Block coordinate descent [14], [15] and *LASSO path* [16] which require high computational cost [17] with convergence issue resulting in large power consumption.

An alternative approach is the optimization under constraint using projection [18], [19]. Note that projecting onto the ℓ_1 norm ball is of linear-time complexity [20], [21]. Unfortunately, these methods generally produce sparse weight matrices, but this sparsity is not structured and thus is not computationally processing efficient. Thus, a structured sparsity is required (i.e. a sparsity able to set a whole set of columns to zero). The $\ell_{1,\infty}$ projection is of particular interest because it is able to set a whole set of columns to zero, instead of spreading zeros as done by the ℓ_1 norm. This makes it particularly interesting for reducing computational cost. Many projection algorithms were proposed [22], [23]. However, the complexity of these algorithms remains an issue. The worst-case time complexity of this algorithm is $\mathcal{O}(nm * \log(nm))$ for a matrix in $\mathbb{R}^{n \times m}$. This complexity is an issue, and to the best of our knowledge, no current publication reports the use of the $\ell_{1,\infty}$ projection for sparsifying large neural networks.

The paper is organized as follows. First, we provide the current state of the art of the $\ell_{1,\infty}$ ball projection. Then, we provide in section III the new bi-level $\ell_{1,\infty}$ projection. In section IV, we apply our bi-level framework to other constraints providing sparsity, such as $\ell_{1,1}$ and $\ell_{1,2}$ constraints. In Section V, we finally compare different projection methods experimentally. First, we provide an experimental analysis of the projection algorithms onto the bi-level projection $\ell_{1,\infty}$ ball. This section shows the benefit of the proposed method, especially for time processing and sparsity. Second, we apply our framework to the classification using a supervised autoencoder on two synthetic datasets and a biological dataset.

II. STATE OF THE ART OF THE $\ell_{1,\infty}$ BALL PROJECTION

In this paper we use the following notations: lowercase Greek symbol for scalars, scalar i,j,c,m,n are indices of vectors and matrices, lowercase for vectors and capital for matrices.

The $\ell_{1,\infty}$ ball projection has shown its efficiency to enforce structured sparsity.[22], [24], [25], [23] and the classical approach is given as follows.

Let $Y \in \mathbb{R}^{n \times m}$ be a real matrix of dimensions $n \times m$, the elements of Y are denoted by $Y_{i,j}$, $i = 1, \dots, n$, $j = 1, \dots, m$. The $\ell_{1,\infty}$ norm of Y is

$$\|Y\|_{1,\infty} := \sum_{j=1}^m \max_{i=1,\dots,n} |Y_{i,j}|. \quad (1)$$

Given a radius $\eta \geq 0$, the goal is to project Y onto the $\ell_{1,\infty}$ norm ball of radius η , denoted by

$$\mathcal{B}_\eta^{1,\infty} := \left\{ X \in \mathbb{R}^{n \times m} : \|X\|_{1,\infty} \leq \eta \right\}. \quad (2)$$

The projection $P_{\mathcal{B}_\eta^{1,\infty}}$ onto $\mathcal{B}_\eta^{1,\infty}$, also noted $P_\eta^{1,\infty}$ in the sequel, is given by:

$$P_\eta^{1,\infty}(Y) = P_{\mathcal{B}_\eta^{1,\infty}}(Y) = \arg \min_{X \in \mathcal{B}_\eta^{1,\infty}} \frac{1}{2} \|X - Y\|_{\mathbb{F}}^2 \quad (3)$$

where $\|\cdot\|_{\mathbb{F}} = \|\cdot\|_{2,2}$ is the Frobenius norm.

Let define the dual $\ell_{\infty,1}$ norm:

$$\|Y\|_{\infty,1} := \max_{j=1,\dots,m} \sum_{i=1}^n |Y_{i,j}|. \quad (4)$$

Given a matrix $Y \in \mathbb{R}^{n \times m}$ and a regularization parameter $\alpha > 0$, the proximity operator of $\alpha \|\cdot\|_{\infty,1}$ is the mapping [26]

$$\text{prox}_{\alpha \|\cdot\|_{\infty,1}} : Y \mapsto \arg \min_{X \in \mathbb{R}^{n \times m}} \frac{1}{2} \|X - Y\|_{\mathbb{F}}^2 + \alpha \|X\|_{\infty,1}. \quad (5)$$

The proximity operator of the dual norm can be easily computed, then, using the Moreau identity [26], [27], [28], [29] is an efficient method for computing the projection onto the $\ell_{1,\infty}$ norm ball:

$$P_{\mathcal{B}_\alpha^{1,\infty}}(Y) = Y - \text{prox}_{\alpha \|\cdot\|_{\infty,1}}(Y) \quad (6)$$

A full description of the projection $P_\eta^{1,\infty}$, using Moreau identity and algorithm to compute it, can be found in [30], [23].

III. A NEW BI-LEVEL $\ell_{1,\infty}$ STRUCTURED PROJECTION

A. A new bi-level projection

In this paper, we propose the following alternative new bi-level method. Let consider a matrix Y with n rows and m columns. Let y_1, \dots, y_m the column vectors of matrix Y . Let $v_\infty = (\|y_1\|_\infty, \dots, \|y_m\|_\infty)$ the row vector composed of the infinity norms of the columns of matrix Y . The bi-level $\ell_{1,\infty}$ projection optimization problem is defined by:

$$BP_\eta^{1,\infty}(Y) = \{X | \forall j, x_j = \arg \min_{x \in \mathcal{B}_{\hat{u}_j}^\infty} \|x - y_j\|_2$$

$$\text{such that } \hat{u} \in \arg \min_{u \in \mathcal{B}_\eta^1} \|u - v_\infty\|_2\} \quad (7)$$

This problem is composed of two problems. The first one, the inner one, is:

$$\hat{u} \in \arg \min_{u \in \mathcal{B}_\eta^1} \|u - v_\infty\|_2 \quad (8)$$

Once the columns of the matrix have been aggregated to a vector v_∞ using the ∞ norm, the problem becomes a usual ℓ_1 ball projection problem. The row vector \hat{u} is given by the following projection of row vector v :

$$\hat{u} \leftarrow P_\eta^1(\|y_1\|_\infty, \dots, \|y_m\|_\infty) \quad (9)$$

Remark III.1. As a contracting property of the P_η^1 projection, we have:

$$\|y_j\|_\infty \geq \hat{u}_j \geq 0 \quad \forall j \in 1, \dots, m \quad (10)$$

These bounds on the u_j hold whatever the norm of the columns y_j .

Then, the second part of the bi-level optimization problem, once the row vector \hat{u} is known, is given by:

$$x_j = \arg \min_{x \in \mathcal{B}_{\hat{u}_j}^\infty} \|x - y_j\|_2 \quad (11)$$

For each column y_j of the original matrix, we compute an estimated column x_j . Each column x_j is optimally computed using the projection on the ℓ_∞ ball of radius \hat{u}_j :

$$x_j \leftarrow P_{\hat{u}_j}^\infty(y_j) \quad \forall j \in 1, \dots, m \quad (12)$$

which can be written as

$$X_{i,j} = \text{sign}(Y_{i,j}) \min(|Y_{i,j}|, \hat{u}_j). \quad (13)$$

Remark III.2. We say that $Y \rightarrow BP_\eta^{1,\infty}$ is a *clipping* operator, and \hat{u} is its clipping threshold.

$$Y_{i,j} - X_{i,j} = \text{sign}(Y_{i,j})(|Y_{i,j}| - \min(|Y_{i,j}|, \hat{u}_j))$$

$$|Y_{i,j} - X_{i,j}| = ||Y_{i,j}| - \min(|Y_{i,j}|, \hat{u}_j)| \quad (14)$$

and then with remark III.1

$$\forall j \quad \max_i |Y_{i,j} - X_{i,j}| = \max_i |Y_{i,j}| - \hat{u}_j = \|y_j\|_\infty - \hat{u}_j \quad (15)$$

Note by 12 that $\hat{u}_j = \|x_j\|_\infty$, so

$$\forall j \quad \|y_j - x_j\|_\infty = \|y_j\|_\infty - \|x_j\|_\infty \quad (16)$$

Algorithm 1 is a possible implementation of BP . It is important to remark that usual bi-level optimization requires many iterations [31], [32] while our model reaches the optimum in one iteration.

Algorithm 1 Bi-level $\ell_{1,\infty}$ projection ($BP_\eta^{1,\infty}(Y)$).

Input: Y, η
 $u \leftarrow P_\eta^1(\|y_1\|_\infty, \dots, \|y_j\|_\infty, \dots, \|y_m\|_\infty)$
for $j \in [1, \dots, m]$ **do**
 $x_j \leftarrow P_{u_j}^\infty(y_j)$
end for
Output: X

B. The $\ell_{1,\infty}$ identity

In the case of $\ell_{1,\infty}$ projection, we needed Moreau's identity to develop the projection algorithm from the "Prox". In the case of our new bilevel $\ell_{1,\infty}$ projection, we have a direct linear-complexity algorithm that does not require Moreau's identity. The aim of this section is to show the respective properties of these two projections. We study a norm of the projected regularized solution versus a norm of the corresponding residual [33]. Recall the classical triangle inequality, which is a consequence of the Cauchy-Schwartz inequality,

$$\|Y - BP_\eta^{1,\infty}(Y)\|_2 + \|BP_\eta^{1,\infty}(Y)\|_2 \geq \|Y\|_2 \quad (17)$$

However, we propose the following norm identity for the bilevel $\ell_{1,\infty}$ projection.

Proposition III.3. *In the case of the $\ell_{1,\infty}$ norm, bilevel projected data and residual are linked by the following relation:*

$$\|Y - BP_\eta^{1,\infty}(Y)\|_{1,\infty} + \|BP_\eta^{1,\infty}(Y)\|_{1,\infty} = \|Y\|_{1,\infty} \quad (18)$$

The proof of equation 18 is readily obtained by summation in j of equation 16.

Remark III.4. Careful examination of the $P_\eta^{1,\infty}$ projector algorithm [30], [23], [22] shows that $P_\eta^{1,\infty}$ is also obtained by a clipping operator, for a different threshold u (See Line 15 of algorithm 1 in [30]) and thus projection verifies 16.

Proposition III.5. *The usual $P_\eta^{1,\infty}$ projection has the following property:*

$$\|Y - P_\eta^{1,\infty}(Y)\|_{1,\infty} + \|P_\eta^{1,\infty}(Y)\|_{1,\infty} = \|Y\|_{1,\infty} \quad (19)$$

The proof of Eq 19 follows the same way as for Eq 18. In fact, identities such as 18 and 19 hold for infinitely many clipping operators. A vector u is a feasible clipping threshold if it satisfies bounds of remark III.1 and sum to η .

Remark III.6. Among all clipping operators, $P_\eta^{1,\infty}$ and $BP_\eta^{1,\infty}$ have the best properties for our purpose. BP has the best structured sparsification effect while P has the best L_2 error, However L_2 error is not more relevant for our purpose than any other norm (for example for the norm, $\ell_{1,\infty}$ BP and P provide the same error).

C. Convergence and Computational complexity

The best computational complexity of the projection of a matrix in \mathbb{R}^{nm} onto the $\ell_{1,\infty}$ ball is usually $O(nm \log(nm))$ [22], [23].

Our bilevel algorithm is split in 2 successive projections. These projections give us a direct solution without iteration, so our algorithm converges in one loop.

The first projection is a ℓ_1 projection applied to the m -dimensional vector of column norms; its complexity is therefore $O(m)$ [20], [21]. The second part is a loop (on the number of columns) of the ℓ_∞ projection, which is implemented with a simple clipping, so its complexity is $O(nm)$. Therefore, the computational complexity of the bi-level projection here is $O(nm)$.

IV. EXTENSION TO OTHER SPARSE STRUCTURED PROJECTIONS

Let recall that there is a close connection both between the proximal operator [34] of a norm and its dual norm, as well as between proximal operators of norms and projection operators onto unit norm balls (pages 187-188, section 6.5 of [35]).

In this section, we extend our bilevel method to the $\ell_{1,1}$ and $\ell_{1,2}$ balls, yielding structured sparsity.

A. Bilevel $\ell_{1,1}$ projection

Let $v_1 = (\|y_1\|_1, \dots, \|y_m\|_1)$ the row vector composed of the ℓ_1 norm of the columns of the matrix Y . We propose to define the $\ell_{1,1}$ bi-level optimization problem:

$$BP_\eta^{1,1}(Y) = \{X | \forall j, x_j = \arg \min_{x \in \mathcal{B}_{u_j}^1} \|x - y_j\|_2$$

$$\text{such that } \hat{u} \in \arg \min_{u \in \mathcal{B}_\eta^1} \|u - v_1\|_2\} \quad (20)$$

A possible implementation of the bi-level $\ell_{1,1}$ is given in Algorithm 2.

Algorithm 2 Bi-level $\ell_{1,1}$ projection. ($BP_\eta^{1,1}(Y)$)

Input: Y, η
 $u \leftarrow P_\eta^1(\|y_1\|_1, \dots, \|y_m\|_1)$
for $j \in [1, \dots, m]$ **do**
 $x_j \leftarrow P_{u_j}^1(y_j)$
end for
Output: X

Consider the $P_{u_j}^1$ projection on the ℓ_1 ball of radius u_j in \mathbb{R}^n :

The projection $x_j = P_{u_j}^1(y_j)$ is obtained by elementwise soft thresholding (proposition 2.2 in [20] or section 6.5.2 in [35]), that is, there exists some positive unique λ_j verifying a critical equation such that:

$$x_j = \max(y_j - \lambda_j, 0) - \max(-y_j - \lambda_j, 0)$$

and

$$|x_j|_1 = u_j$$

so

$$Y_{i,j} - X_{i,j} = \text{sign}(Y_{i,j}) |Y_{i,j}| - \max(|Y_{i,j}| - \lambda_j, 0) \quad (21)$$

and

$$|Y_{i,j} - X_{i,j}| = |Y_{i,j}| - \max(|Y_{i,j}| - \lambda_j, 0)$$

$$= |Y_{i,j}| - \max(|Y_{i,j}| - \lambda_j, 0) \quad (22)$$

$$= |Y_{i,j}| - |X_{i,j}|$$

which can be written, by summing on i :

$$\forall j \quad \|y_j - x_j\|_1 = \|y_j\|_1 - \|x_j\|_1 \quad (23)$$

By direct summation on j of equation 23, we have:

Proposition IV.1. *The bilevel $\ell_{1,1}$ projection satisfies the following identity.*

$$\|Y - BP_\eta^{1,1}(Y)\|_{1,1} + \|BP_\eta^{1,1}(Y)\|_{1,1} = \|Y\|_{1,1} \quad (24)$$

B. Bilevel $\ell_{1,2}$ projection.

Let $v_2 = (\|y_1\|_2, \dots, \|y_m\|_2)$ the row vector composed of the ℓ_2 norm of the columns of the matrix Y . The $\ell_{1,2}$ bi-level optimization problem be:

$$BP_\eta^{1,2}(Y) = \{X | \forall i, x_i = \arg \min_{x_i \in \mathcal{B}_{u_i}^2} \|x_i - y_i\|_2$$

$$\text{such that } \hat{u} \in \arg \min_{u \in \mathcal{B}_\eta^1} \|u - v_2\|_2\} \quad (25)$$

Similarly, the bi-level projection algorithms for $\ell_{1,2}$ is given by algorithm 3.

Algorithm 3 Bi-level $\ell_{1,2}$ projection. ($BP_\eta^{1,2}(Y)$)

Input: Y, η
 $u \leftarrow P_\eta^1(\|y_1\|_2, \dots, \|y_m\|_2)$
for $j \in [1, \dots, m]$ **do**
 $x_j \leftarrow P_{u_j}^2(y_j)$
end for
Output: X

Let, $x_j = P_{u_j}^2(y_j)$ then, $x_j = \frac{u_j}{\|y_j\|_2} y_j$ (section 6.5.1 in [35]) and so $y_j - x_j = (1 - \frac{u_j}{\|y_j\|_2}) y_j$ then

$$\forall j \quad \|y_j - x_j\|_2 = \|y_j\|_2 - u_j = \|y_j\|_2 - \|x_j\|_2 \quad (26)$$

which by a direct summation on j leads to

Proposition IV.2. *The bilevel $\ell_{1,2}$ projection satisfies the following identity.*

$$\|Y - BP_\eta^{1,2}(Y)\|_{1,2} + \|BP_\eta^{1,2}(Y)\|_{1,2} = \|Y\|_{1,2} \quad (27)$$

V. EXPERIMENTAL RESULTS

A. Benchmark times using PyTorch C++ extension using a MacBook Laptop with an i9 processor; Comparison with the best actual projection method

The experiments were run on a laptop with a I9 processor having 32 GB of memory. The state of the art on such is pretty large, starting with [22] who proposed the first algorithm, the Newton-based root-finding method and column elimination method [24], [23], and the recent paper of *Chu et al.* [25] which outperforms all the other state-of-the-art methods. We compared our bi-level method against all the existing projection algorithms, and our algorithm is faster in all the scenarios. The algorithm from [22] is on average 30 times slower than our algorithm, note that this factor growth logarithmically with data size. As shown in [25], the best actual algorithm is proposed by *Chu et al.* which uses a semi-smooth Newton algorithm for the projection. That is why we focus our presentation on comparing against this particular method. We use the C++ implementation provided by the authors and the PyTorch C++ implementation of our bi-level $\ell_{1,\infty}$ method is based on fast ℓ_1 projection algorithms of [20], [21] which are of linear complexity. The code of all the compared algorithms is available online¹.

Figure 1 shows the running time as a function of the matrix size. Here the radius has been fixed to $\eta = 1$. The fitting of a linear curve (red) on the data (blue) shows that the running time of our bilevel $\ell_{1,\infty}$ projection is linear with the number of features and the number of samples. The fitting of a $n \log n$ curve (green) on the data (orange) shows that the running time of the usual $\ell_{1,\infty}$ projection grows as $n \log n$ with the number of features and the number of samples. Moreover, the slope of the usual projection algorithm is greater by a factor 2.5 on both graphs than the slope of the bilevel algorithm.

¹<https://github.com/memo-p/projection>

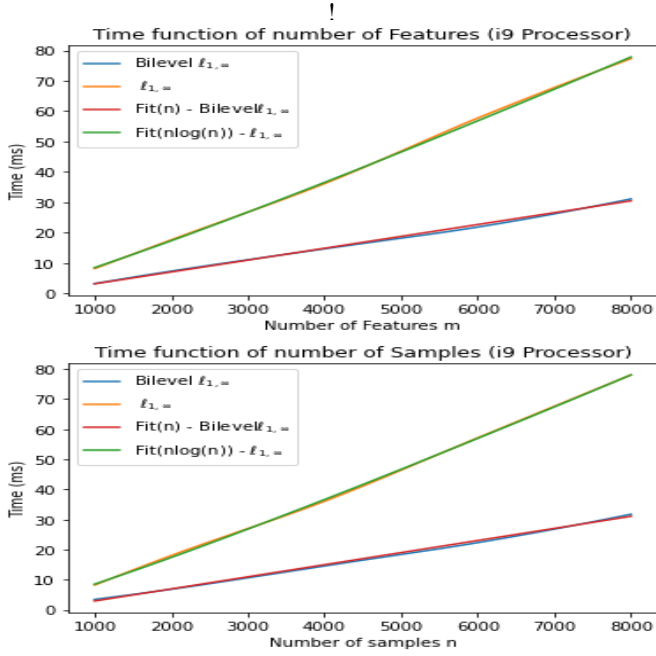


Fig. 1. Processing time using C++ as a function of the number of features $n = 1000$ samples (top) and Samples $m = 1000$ features (bottom): bi-level projection method versus *Chu et al.* method.

Figure 2 shows that all bilevel algorithms have the same slopes for time versus feature or sample number.

Note that PyTorch c++ extension is 20 times faster than the standard PyTorch implementation.

B. Benchmark of Identity Proposition

We generate two artificial biological datasets to benchmark our bi level projection using the *make_classification* utility from *scikit-learn*. We generate $n = 1,000$ samples with a number of $m = 1,000$ features. The first one with 64 (data-64) informative features and the second (data-16) with 16 informative features

We provide the experimental proof of the proposition and the sparsity score in %: number of columns or features set to zero.

Fig 3 shows that the two curves (Bilevel and usual $\ell_{1,\infty}$ projections) and the η parameter are perfectly coincident, and perfectly linear as expected by the identity equation (18).

Remark V.1. The identity equations hold only if the norm is similar to the projection. Figure 4 shows that the identity equation is not true when using Bilevel and usual $\ell_{1,\infty}$ projections and the classical $\ell_{2,2}$ norm. Usual projection $\ell_{1,\infty}$ has the lower $\ell_{2,2}$ error. However, $\ell_{2,2}$ error is not more relevant for our purpose than any other norm.

Cum-Sparsity (%)	bilevel $\ell_{1,\infty}$	bilevel $\ell_{1,1}$	bilevel $\ell_{1,2}$	$\ell_{1,\infty}$
datas-64	5.36	4.714	4.705	1.872
data-16	1.99	1.09	1.07	0.419

TABLE I

COMPARISON OF SPARSITY FOR TWO DATASETS WITH DIFFERENT INFORMATIVE FEATURES, WHERE CUM-SPARSITY (%) IS THE SUM OF SPARSITY OVER THE TEST SETS

Table I shows that our bilevel $\ell_{1,\infty}$ projection outperforms sparsity of the usual $\ell_{1,\infty}$ projection. Although the sparsity curves look very close. Table I shows that bilevel $\ell_{1,1}$ projection is slightly more sparse than bilevel $\ell_{1,2}$.

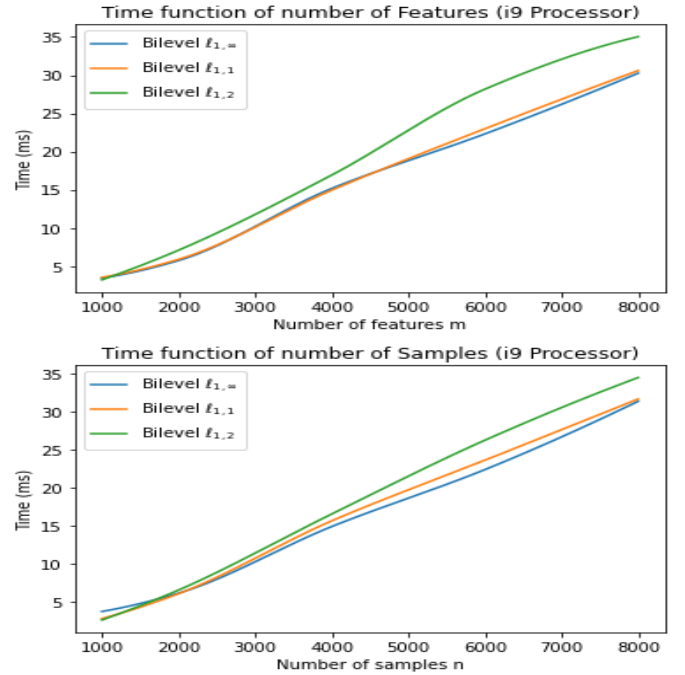


Fig. 2. Processing time using C++ as a function of the number of features (Top), and samples (bottom)

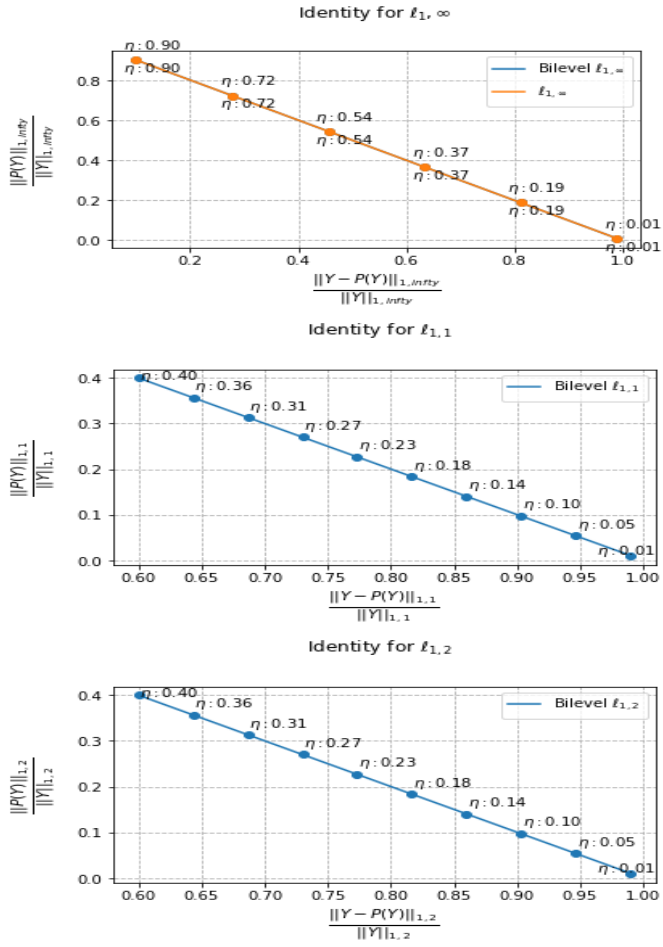


Fig. 3. Identity norm comparison Top: the Bilevel $\ell_{1,\infty}$ versus classical, Middle: Bilevel $\ell_{1,1}$, bottom: Bilevel $\ell_{1,2}$ projection.

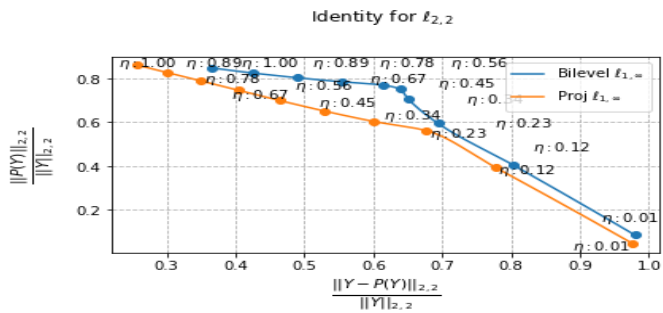


Fig. 4. Bilevel $\ell_{1,\infty}$ projection and usual $\ell_{1,\infty}$ projection with $\ell_{2,2}$ norm.

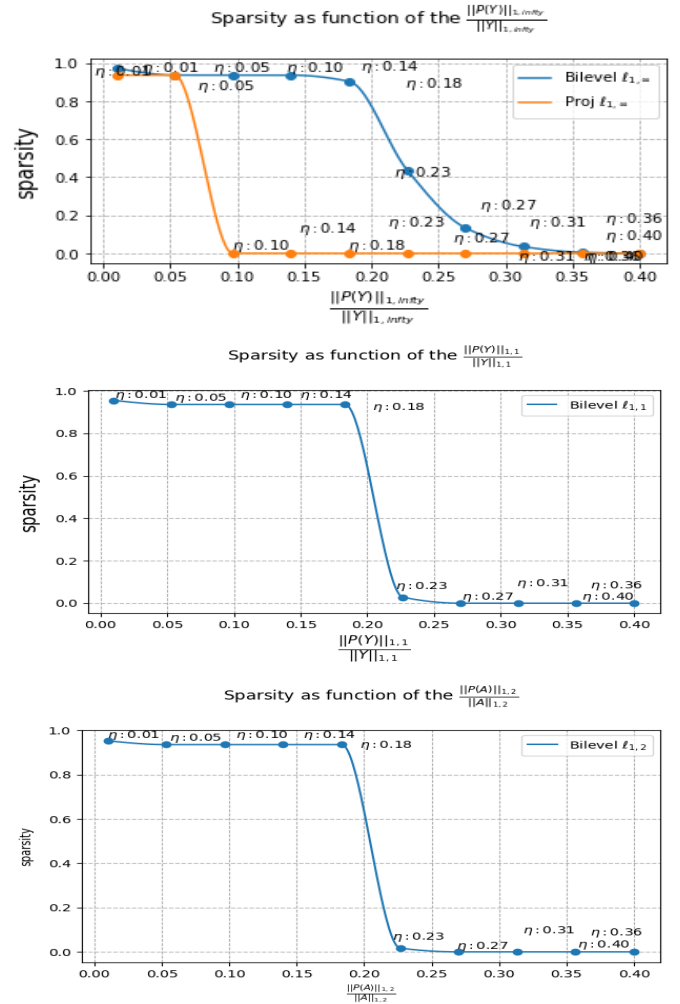


Fig. 5. 64 informative features Sparsity Top: the Bilevel $\ell_{1,\infty}$, Middle: Bilevel $\ell_{1,1}$, bottom: Bilevel $\ell_{1,2}$ projection

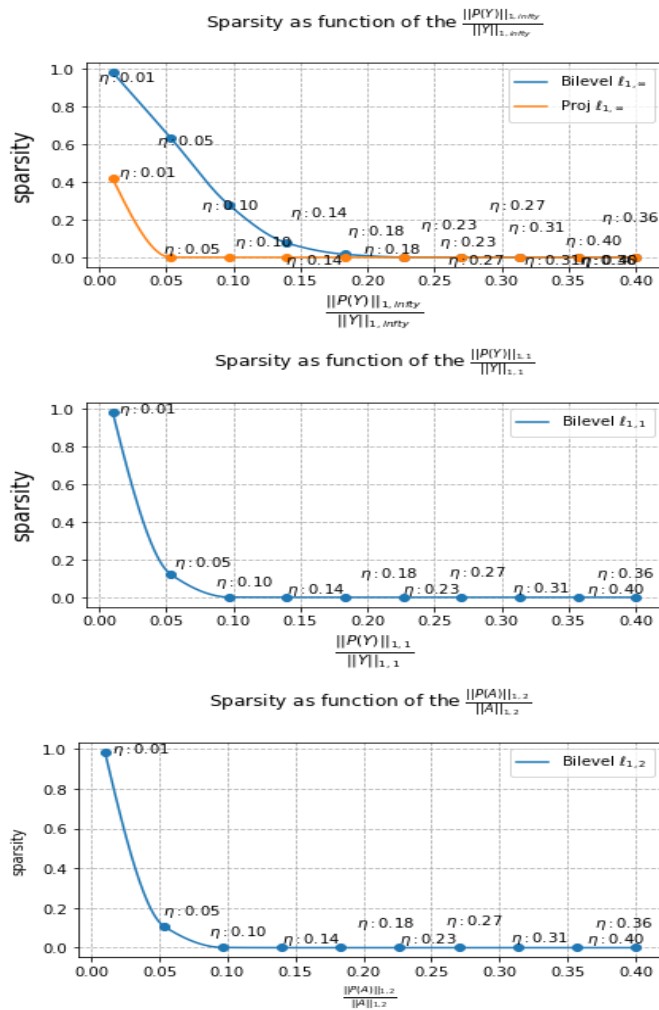


Fig. 6. 16 informative features. Sparsity: Top the Bilevel $\ell_{1,\infty}$, Middle: Bilevel $\ell_{1,1}$, bottom: Bilevel $\ell_{1,2}$ projection

The code is available online ²

C. Experimental results on classification and feature selection using a supervised autoencoder neural network

1) *Supervised Autoencoder (SAE) framework*: Autoencoders were introduced within the field of neural networks decades ago, their most efficient application being dimensionality reduction [36]. Autoencoders were used in application ranging from unsupervised deep-clustering [37], [38] to supervised learning, adding a classification loss in order to improve classification performance [39], [40]. In this paper, we use a supervised autoencoder with the cross entropy as the added classification loss.

Let X be the concatenated raw data matrix ($n \times m$) (n is the number of samples (cells) and m the number of genes). Let \hat{X} the reconstructed data and W the weights of the neural network. Let Z the encoded latent space. Note that the dimension of the latent space k corresponds to the number of classes.

The goal is to learn the network weights, W minimizing the total loss. In order to sparsify the neural network, we propose to use the different bi-level projection methods as a constraint to enforce sparsity in our model. The global criterion is:

$$\underset{W}{\text{minimize}} \quad \phi(X, Y) \quad \text{subject to} \quad BP^{1,\infty}(W) \leq \eta \quad (28)$$

where $\phi(X, Y) = \alpha \cdot \psi(X, \hat{X}) + \mathcal{H}(Y, Z)$. We use the robust Smooth ℓ_1 (Huber) Loss [41] as the reconstruction loss ψ . Parameter α is a linear combination factor used to define the final loss. We compute the mask by using the various bilevel projection methods, and we use the double descent algorithm [42], [43] for minimizing the criterion 28. We implemented our SAE method using the PyTorch framework for the model, optimizer, schedulers and loss functions. We use a fully connected neural network with only one hidden layer (dimension 100) and a latent layer of dimension $k = 2$. We chose the ADAM optimizer [44], as the standard optimizer in PyTorch. We use the smooth SiLU activation function.

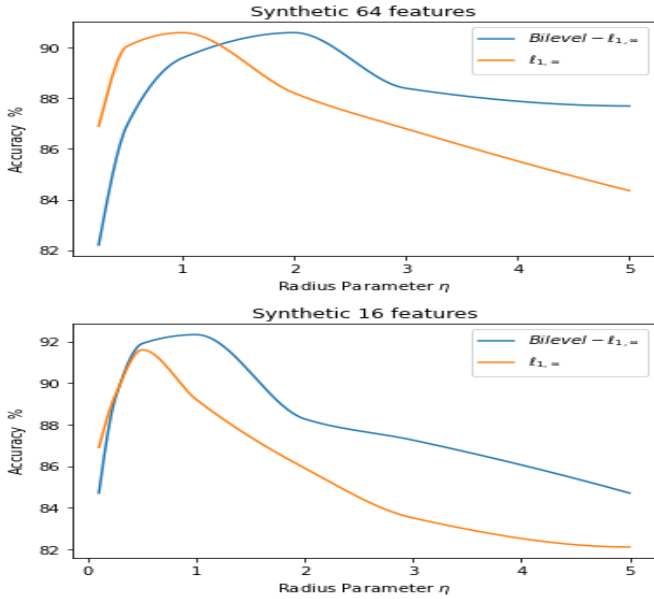


Fig. 7. Accuracy as a function of the radius parameter η ; Top, 64 informative features :Bottom 16 informative features

2) *Experimental accuracy results on autoencoder neural networks* : Figure 7 shows the impact of the radius (η) on a synthetic

dataset using the bilevel projection versus the usual projection. It can be seen that the best accuracy is obtained for $\eta = 0.5$ for $\ell_{1,\infty}$ and for $\eta = 1$ for the bilevel $\ell_{1,\infty}$ projection. Moreover, the accuracy is more robust to parameter η for the bilevel projection. Table II shows accuracy classification for 64 informative features. The baseline is an implementation that does not process any projection. Compared to the baseline, the SAE using the $\ell_{1,\infty}$ projection improves the accuracy by 10.3%.

Table III shows that accuracy results for 16 informative features of the bilevel $\ell_{1,\infty}$ and classical $\ell_{1,\infty}$ is slightly better.

From Tables II on synthetic dataset it can be seen that the best accuracy is obtained for $\eta = 0.5$ for $\ell_{1,\infty}$ and for $\eta = 1$. for the bilevel $\ell_{1,\infty}$ projection. Maximum accuracy of both method are similar. However, sparsity and computation time are better for bilevel $\ell_{1,\infty}$ than for regular $\ell_{1,\infty}$.

Synthetic 64	Baseline	$\ell_{1,\infty}$	bilevel $\ell_{1,\infty}$
Best Radius	-	1	2.0
Accuracy %	80.3 \pm 1.8	90.6.6 \pm 2.85	90.6 \pm 1.24

TABLE II
SYNTHETIC DATASET 64 FEATURES. SiLU ACTIVATION, ACCURACY : COMPARISON OF $\ell_{1,\infty}$ AND BI-LEVEL $\ell_{1,\infty}$.

Synthetic 16	Baseline	$\ell_{1,\infty}$	bilevel $\ell_{1,\infty}$
Best Radius	-	0.5	1.0
Accuracy %	74.6 \pm 2.2	91.6 \pm 3.3	92.36 \pm 1.9

TABLE III
SYNTHETIC DATASET 16 FEATURES. SiLU ACTIVATION, ACCURACY : COMPARISON OF $\ell_{1,\infty}$ AND BI-LEVEL $\ell_{1,\infty}$.

We consider the HIF2 dataset, now one of the real dataset from the study of in single-cell CRISPRi screening [45]. This HIF2 dataset is composed of 779 cells and 10,000 features (genes) [45]. Figure 8

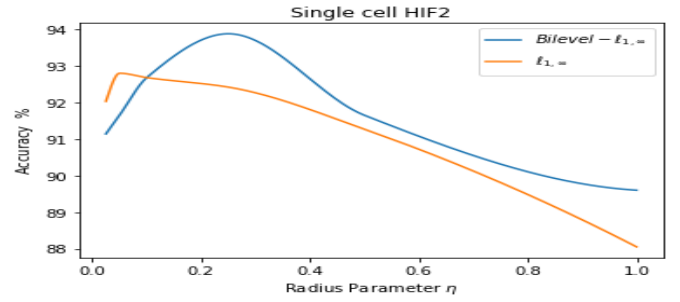


Fig. 8. Accuracy as a function of the radius parameter η ; HIF2 dataset

shows the curve of the accuracy versus the radius (η) on HIF2 dataset. We can see that the curve is more well shaped using by the bilevel projection.

Real data HIF2	Baseline	$\ell_{1,\infty}$	bilevel $\ell_{1,\infty}$
Best Radius	-	0.1	0.25
Accuracy %	84.6 \pm 1.2	92.68 \pm 1.85	93.88 \pm 1.8

TABLE IV
HIF2 DATASET SiLU ACTIVATION, ACCURACY : COMPARISON OF $\ell_{1,\infty}$ AND BI-LEVEL $\ell_{1,\infty}$.

Again, tableIV on this HIF2 real dataset, compared to the baseline the SAE using the bilevel $\ell_{1,\infty}$ projection improves the accuracy by 10% similarly to the results on the synthetic data-64. Moreover, on this real dataset, our bilevel projection outperforms the usual $\ell_{1,\infty}$ projection by 1.2%.

²<https://github.com/MichelBarlaud/SAE-Supervised-Autoencoder-Omics>

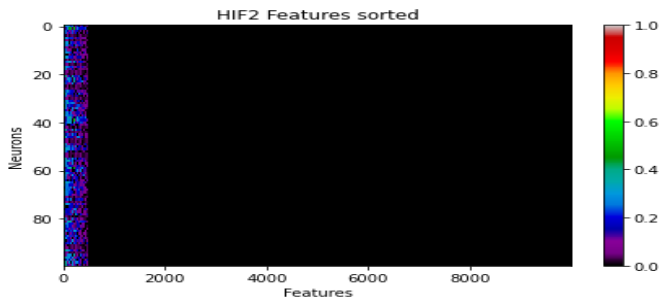


Fig. 9. Weights of the first layer of the Fully connected Neural network

Constraint optimization generally produces networks with random sparse connectivity, i.e. sparse weight matrices. They only reduce the memory cost but not the computational cost, since they result in networks with sparse weight matrices. Decreasing both memory and computational requirements can however be achieved by suppressing columns (features) instead of weights. Figure 9 shows that our bilevel $\ell_{1,\infty}$ suppress columns and thus efficiently reduces the computational cost.

VI. DISCUSSION

A first application of our bilevel $\ell_{1,\infty}$ projection is feature selection in biology [45].

Second, our bilevel method can be extended straight forward to tensor and convolutional neural networks. The $\ell_{1,\infty}$ projection has been successfully applied to the sparsification of autoencoders using convolutional neural networks with a memory reduction of 84 and a reduction in computational cost of 40 without visual image degradation [46] in the new image compression standard JPEG AI [47].

According to estimates, by 2040 Artificial Intelligence (AI) server farms may account for 14 of all global carbon emissions in the world. Third, another application of our method can be the sparsification of attention matrices of a transformer architecture [48] used in AI software.

VII. CONCLUSION

Although many projection algorithms were proposed for the projection of the $\ell_{1,\infty}$ norm, complexity of these algorithms remain an issue. The worst-case time complexity of these algorithms is $\mathcal{O}(n \times m \times \log(n \times m))$ for a matrix in $\mathbb{R}^{n \times m}$. In order to cope with this complexity issue, we have proposed a new bi-level projection method. The main motivation of our work is the direct independent splitting made by the bi-level optimization, which takes into account the structured sparsity requirement. We showed that the theoretical computational cost of our new bi-level method is only $\mathcal{O}(n \times m)$ for a matrix in $\mathbb{R}^{n \times m}$ and thus improves by $\mathcal{O}(\log(nm))$ times classical algorithms. Note that this improvement can be huge for a large dataset. Experiments on synthetic and a real data show that our bi-level method is faster than the actual fastest algorithm. Sparsity of our bi-level $\ell_{1,\infty}$ projection outperforms other bi-level projections $\ell_{1,1}$ and $\ell_{1,2}$.

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